Exact inhomogeneous Einstein-Maxwell-Dilaton cosmologies

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We present solution generating techniques which permit to construct exact inhomogeneous and anisotropic cosmological solutions to a four-dimensional low energy limit of string theory containing non-minimally interacting electromagnetic and dilaton fields. Some explicit homogeneous and inhomogeneous cosmological solutions are constructed. For example, inhomogeneous exact solutions presenting Gowdy - type EMD universe are obtained. The asymptotic behaviour of the solutions is investigated. The asymptotic form of the metric near the initial singularity has a spatially varying Kasner form. The character of the space-time singularities is discussed. The late evolution of the solutions is described by a background homogeneous and anisotropic universe filled with weakly interacting gravitational, dilatonic and electromagnetic waves.

I. INTRODUCTION

In the last decade string cosmologies attracted large amount of interest (see e.g. [1] and references therein). In the traditional approach the present (low energy) string cosmology is in fact the classical cosmology where general relativity is generalized by including additional (in most cases) massless scalar fields. In general, one expects that the inclusion of these extra matter degrees of freedom with the corresponding physical interpretation, may somehow resolve the long standing problems in cosmology. A recent interesting and promising development in this direction is the so called *pre-big bang* scenario [2]. In the framework of this scenario one assumes that the initial state of the universe is characterized by small string coupling and small curvature. This leads to an inflationary phase for sufficiently homogeneous initial conditions. There are, however, no natural reasons for the early universe not be inhomogeneous. At present it is not clear how the large inhomogeneities may influence the pre-big bang scenario. In a more general setting, the question of whether our universe (which looks now homogeneous and isotropic) may arise out of generic initial data still lacks a complete answer in both general relativistic and string cosmology.

That is why the construction and study of exact inhomogeneous and anisotropic string-cosmological solutions are still of great importance.

The inhomogeneous string cosmologies have been studied by a number of authors. In [3], Barrow and Kunze have presented inhomogeneous and anisotropic cosmological solutions of a low energy string theory containing dilaton and axion fields when the space-time metric possesses cylindrical symmetry . Their solutions describe ever-expanding universes with an initial curvature singularity. The asymptotic form of the solutions near the initial singularity has a spatially varying Kasner-like form.

In [4], Fienstein, Lazkoz, and Vazquez - Mozo have presented an algorithm for constructing exact solutions in string cosmology for heterotic and type - IIB superstrings in four dimensions. They have also presented and discussed some properties of an inhomogeneous string cosmology with S^3 topology of the spatial sections. Clancy et al. [5] have derived families of anisotropic and inhomogeneous string cosmologies containing nontrivial dilaton and axion fields by applying the global symmetries of the string-effective action to a generalized Eistein-Rosen metric. Lazkoz [6] has presented an algorithm for generating families of inhomogeneous space-times with a massless scalar field. New solutions to Einstein - massless scalar field equations having single isometry, have been generated in [6] by breaking the homogeneity of massless scalar field G_2-models along one direction.

It should be noted, however, that the inhomogeneous cosmologies in the framework of general relativity with certain physical fields as a matter source have been investigated long before the time of the string cosmology. Exact inhomogeneous vacuum Einstein cosmologies with $S^1 \times S^1 \times S^1$, $S^2 \times S^1$ and S^3 topology of spatial sections have been found by Gowdy [7] and studied afterwards by Berger [8] and Misner [9]. Exact stiff perfect fluid inhomogeneous cosmologies have been studied by Wainwright, Ince and Marshman [10]. Later Charach [11] and then Charach and Malin [12] (see also [13]) have found and studied exact inhomogeneous cosmological solutions to the Einstein equations with an electromagnetic and minimally coupled scalar field. These solutions have been interpreted as an inhomogeneous universe filled with gravitational, scalar and electromagnetic waves.

The purpose of this paper is to present sufficiently general solution generating techniques which permit to construct exact inhomogeneous and anisotropic solutions to the equations of low energy string theory containing non-minimally coupled dilaton and electromagnetic fields -

the so called Einstein-Maxwell-dilaton (EMD) gravity. Examples of both homogeneous and inhomogeneous exact solutions will be also presented and their asymptotics will be investigated.

Besides the fact that the EMD cosmologies are interesting in their own, there are at least two more motivations for considering the EMD cosmologies. First, a long standing question in cosmology is the existence of a primordial magnetic field. The string theory prediction of the non-minimal coupling of the dilaton to the Maxwell field gives an opportunity to study the problem in a more general dynamical context. Recently, in the homogeneous case this question has been addressed in [14].

The second motivation is to investigate the character of initial cosmological singularities in the presence of matter fields arising from the low energy super-string theory. According to the Belinskii-Khalatnikov-Lifhsitz (BKL) conjecture the dynamics of nearby observers would decouple near the singularity for different spatial points. A vacuum spatially homogeneous space-time singularity (IX Bianchi type) is described by BKL as an infinite sequence of Kasner epochs ("oscillatory or mixmaster behaviour") [15], [16]. BKL further speculated that a generic singularity should exhibit such a local oscillatory behaviour.

Interesting special cases are the so-called asymptotic velocity-term dominated (AVTD) singularities [17]. Their characteristic feature is the fact that the spatial derivative terms in the field equations are negligible sufficiently close to the singularities. Such singularities are described by only one Kasner epoch, not by an infinite sequence of Kasner epochs i.e. an oscillatory behaviour does not exist. We think that it is important to study the behaviour of the singularities in the presence of matter fields coming form the super-string theory. In particular, it is interesting to investigate the influence of non-minimal exponential coupling of the dilaton to the $U(1)$-field on the character of the singularities. As it has been shown in [18], [19] the minimally coupled scalar field can suppress the oscillatory behaviour.

Very recently the influence of the exponential dilaton-electromagnetic coupling on the character of initial singularities has been studied by Narita, Torii and Maeda in [20]. These authors have considered T^3 Gowdy cosmologies in Einstein-Maxwell-dilaton-axion system and have shown using the Fushian algorithm that space-times in general have asymptotic velocity-term dominated singularities. Their results mean that the exponential coupling of the dilaton to the Maxwell field does not change the nature of the singularity. It should be also noted that the exact solutions found in the present work lead to the same conclusion.

Although, in the present paper we shall not consider the BKL conjecture in details, we believe that the present work may serve as a good ground for further studying of the above mentioned questions in the framework of Einstein - Maxwell-dilaton-axion gravity.

II. CONSTRUCTING SOLUTIONS WITH ONE KILLING VECTOR

We consider EMD-gravity described by the action

$$\mathcal{A} = \frac{1}{16\pi} \int \left(\star R - 2d\varphi \wedge \star d\varphi - 2e^{-2\varphi} F \wedge \star F \right)$$

where $\star$ is Hodge dual with respect to the space-time metric g, R is the Ricci scalar curvature, φ is the dilaton field and $F = dA$ is the Maxwell two form.

First we will examine the case of space-time admitting one space-like, hyper-surface orthogonal Killing vector X. Physically this situation corresponds to a universe with homogeneity broken along two space directions. In the presence of a Killing vector the Bianchi identity $dF = 0$ and the Maxwell equations $d \star e^{-2\varphi} F = 0$ give rise to the following local potentials Ψ_e and Ψ_m:

$$d\Psi_e = -i_X F \quad , \quad d\Psi_m = e^{-2\varphi} i_X \star F.$$

Here i_X is the interior product of an arbitrary form with respect to X.

The hyper-surface orthogonal Killing vector X naturally determines a three dimensional space-time submanifold Σ with metric [21]

$$P = \mathcal{N}g - X \otimes X$$

where $\mathcal{N} = g(X, X)$ is the norm of the Killing vector.

We note that throughout the text we denote the Killing fields and their naturally corresponding one-forms by the same symbols.

The projection on Σ of the Einstein equations reads

$$\hat{\mathcal{R}} = 2d\varphi \otimes d\varphi + \frac{1}{2\mathcal{N}^2} d\mathcal{N} \otimes d\mathcal{N} +$$
$$\frac{2}{\mathcal{N}} \left(e^{-2\varphi} d\Psi_e \otimes d\Psi_e + e^{2\varphi} d\Psi_m \otimes d\Psi_m \right) \quad (1)$$

where $\hat{\mathcal{R}}$ is Ricci tensor with respect to the projection metric P. The Maxwell equations take the form

$$d^\dagger \left(e^{-2\varphi} \mathcal{N}^{-1} d\Psi_e \right) = 0 \quad , \quad d^\dagger \left(e^{2\varphi} \mathcal{N}^{-1} d\Psi_m \right) = 0. \quad (2)$$

Here $d^\dagger = \star d \star$ is the co-derivative operator.

The projection of the Einstein equations along to the Killing vector gives

$$\mathcal{N} \Box \mathcal{N} - g(d\mathcal{N}, d\mathcal{N}) =$$
$$-2\mathcal{N} e^{-2\varphi} \left(e^{-2\varphi} g(d\Psi_e, d\Psi_e) + e^{2\varphi} g(d\Psi_m, d\Psi_m) \right). \quad (3)$$

Finally, the vanishing of the twist $\mathcal{T} = \frac{1}{2} \star dX$ of the Killing field leads to the constraint

$$d\Psi_e \wedge d\Psi_m = 0. \quad (4)$$

The equations (1), (2), (3) and (4) are equivalent to the EMD - gravity equations in the presence of a spice-like, hyper-surface orthogonal Killing vector.

The constraint (4) implies that the one-forms $d\Psi_m$ and $d\Psi_e$ are proportional i.e. there exists a function $\mathcal{F}$ such that $d\Psi_m = \mathcal{F}\, d\Psi_e$. Here we will consider the simplest case in which one of the potentials Ψ vanishes, say $\Psi_e = 0$.

In the case $\Psi_e = 0$ the full system dimensionally reduced equations can be obtained from the following action

$$
\mathcal{A}_{\mathcal{R}} = \frac{1}{16\pi} \int \sqrt{-\det P} \left(\mathcal{R} - \frac{P^{ij}}{2\mathcal{N}^2} D_i \mathcal{N} D_j \mathcal{N} \right.
$$
$$
\left. - P^{ij} D_i \varphi D_j \varphi - 2e^{2\varphi} \mathcal{N}^{-1} P^{ij} D_i \Psi_m D_j \Psi_m \right). \qquad (5)
$$

Now we introduce the following symmetric matrix $S \in GL^+(2, R)$:

$$
S = \begin{pmatrix} \mathcal{N} + 2\Psi_m^2 e^{2\varphi} & \sqrt{2}\Psi_m e^{2\varphi} \\ \sqrt{2}\Psi_m e^{2\varphi} & e^{2\varphi} \end{pmatrix}.
$$

Making use of the matrix S, we may write the action (5) in the form

$$
\mathcal{A}_{\mathcal{R}} = \frac{1}{16\pi} \int \sqrt{-\det P} \left(\mathcal{R} - \frac{1}{2} P^{ij} Tr(D_i S D_j S^{-1}) \right). \qquad (6)
$$

The action (6) has $GL(2, R)$ as a group of global symmetry for fixed projection metric P. Explicitly the group $GL(2, R)$ acts as follows :

$$
S \to A S A^T
$$

where $A \in GL(2, R)$.

Note, that in the case when one of the electromagnetic potentials vanishes the dimensionally reduced EMD equations can be viewed as three-dimensional Einstein gravity coupled to a nonlinear σ - model on the factor $GL(2, R)/O(2, R)$.

The $GL(2, R)$ symmetry may be employed for generating new exact solutions from known ones. In particular, it will be very useful to employ the $GL(2, R)$ symmetry to generate exact solutions with nontrivial electromagnetic field from any given solution to the vacuum field equations (i.e. pure Einstein equations) or the dilaton-vacuum equations (Einstein equations plus a dilaton field).

For example, starting with a dilaton-vacuum seed solution

$$
S_{vd} = \begin{pmatrix} \mathcal{N}_{vd} & 0 \\ 0 & e^{2\varphi_{vd}} \end{pmatrix},
$$

the nonlinear action of $GL(2, R)$ gives

$$
\mathcal{N} = (\det A)^2 \frac{\mathcal{N}_{vd}\, e^{2\varphi_{vd}}}{c^2 \mathcal{N}_{vd} + d^2 e^{2\varphi_{vd}}},
$$

$$
e^{2\varphi} = c^2 \mathcal{N}_{vd} + d^2 e^{2\varphi_{vd}},
$$

$$
\Psi_m = \frac{1}{\sqrt{2}} \frac{ac\mathcal{N}_{vd} + bde^{2\varphi_{vd}}}{c^2 \mathcal{N}_{vd} + d^2 e^{2\varphi_{vd}}}
$$

where

$$
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
$$

It is worth noting, however, that some of the $GL(2, R)$-transformations are pure electromagnetic gauge or rescaling of the solutions. Only the $O(2, R)$-transformations lead to essentially new solutions.

Exact solutions to the EMD gravity equations can be constructed from vacuum solutions by the method described in [22], too. In order to obtain solutions by this method one assumes that the matrix S depends on space-time coordinates through a harmonic potential Ω ($D^i D_i \Omega = 0$). Requiring then the constraint

$$
-\frac{1}{4} Tr \left(\frac{dS}{d\Omega} \frac{dS^{-1}}{d\Omega} \right) = 1
$$

to be satisfied, the EMD equations are reduced to the vacuum Einstein equations

$$
\hat{\mathcal{R}}_{ij} = 2 D_i \Omega\, D_j \Omega\,,
$$
$$
D^i D_i \Omega = 0
$$

and a separated matrix equation

$$
\frac{d}{d\Omega} \left(S^{-1} \frac{dS}{d\Omega} \right) = 0\,.
$$

Therefore, for every solution to the vacuum Einstein equations, the solutions of the matrix equation give classes of exact EMD gravity solutions (see for details [22]).

It should be noted that the seed solutions used for the construction of exact EMD solutions via the above described methods, must admit at least one Killing vector. The case with two commuting hyper-surface orthogonal Killing vectors will be considered in the following section.

III. CONSTRUCTING SOLUTIONS WITH TWO KILLING VECTORS

A. General equations

In this section we consider the EMD gravity equations when the space-time admits two commuting space-like, hyper-surface orthogonal Killing vectors X and Y. In this case the metric can be written in the form

$$ds^2 = e^{2h-2p}(dz^2 - d\xi^2) + e^{2p}dx^2 + \varrho \, e^{-2p}dy^2 \qquad (7)$$

where h, p and ϱ are unknown functions of z and ξ only. Therefore the Killing vectors are given by $X = \frac{\partial}{\partial x}$ $Y = \frac{\partial}{\partial y}$.

The physical properties of the metric (7) depend on the gradient of the norm ϱ of the two-form $X \wedge Y$. The case corresponding to $\partial_\mu \varrho$ being globally space-like or null describes cylindrical and plane gravitational waves. Metrics where the sign of $\partial_\mu \varrho \, \partial^\mu \varrho$ can change, describe colliding plane waves or cosmological models with space-like and time-like singularities. Metrics with globally time-like $\partial_\mu \varrho$ describe cosmological models with space-like singularities.

In the present paper we shall consider only the globally time-like case $\varrho = \xi^2$. It should be noted that the choice $\varrho = \xi^2$ is dynamically consistent with the EMD gravity equations because the Ricci tensor $\hat{R}$ satisfies

$$g(Y,Y)\hat{R}(X,X) + g(X,X)\hat{R}(Y,Y) = 0.$$

In the presence of a second Killing vector the dimensional reduction can be further continued. Without going into details we directly present the dimensionally reduced equations

$$\partial_\xi^2 p + \frac{1}{\xi}\partial_\xi p - \partial_z^2 p = e^{-2\varphi-2p}\left((\partial_z \Psi_e)^2 - (\partial_\xi \Psi_e)^2\right)$$
$$+ \; e^{2\varphi-2p}\left((\partial_z \Psi_m)^2 - (\partial_\xi \Psi_m)^2\right), \quad (8)$$

$$\partial_\xi^2 \varphi + \frac{1}{\xi}\partial_\xi \varphi - \partial_z^2 \varphi = e^{-2\varphi-2p}\left((\partial_z \Psi_e)^2 - (\partial_\xi \Psi_e)^2\right)$$
$$- \; e^{2\varphi-2p}\left((\partial_z \Psi_m)^2 - (\partial_\xi \Psi_m)^2\right), \quad (9)$$

$$\partial_\xi \Psi_e \partial_z \Psi_m = \partial_z \Psi_e \partial_\xi \Psi_m, \qquad (10)$$

$$\partial_\xi \left(\xi e^{-2\varphi-2p}\partial_\xi \Psi_e\right) - \partial_z \left(\xi e^{-2\varphi-2p}\partial_z \Psi_e\right) = 0, \quad (11)$$

$$\partial_\xi \left(\xi e^{2\varphi-2p}\partial_\xi \Psi_m\right) - \partial_z \left(\xi e^{2\varphi-2p}\partial_z \Psi_m\right) = 0, \qquad (12)$$

$$\frac{1}{\xi}\partial_\xi h = (\partial_\xi \varphi)^2 + (\partial_z \varphi)^2 + (\partial_\xi p)^2 + (\partial_z p)^2 +$$
$$e^{-2\varphi-2p}\left((\partial_\xi \Psi_e)^2 + (\partial_\xi \Psi_e)^2\right) +$$
$$e^{-2\varphi-2p}\left((\partial_\xi \Psi_m)^2 + (\partial_\xi \Psi_m)^2\right), \qquad (13)$$

$$\frac{1}{\xi}\partial_z h = 2\partial_\xi \varphi \partial_z \varphi \; + 2\partial_\xi p \partial_z p +$$
$$2e^{-2\varphi-2p}\partial_\xi \Psi_e \partial_z \Psi_e + \; 2e^{2\varphi-2p}\partial_\xi \Psi_m \partial_z \Psi_m \, . \qquad (14)$$

Here we should make some comments. We have written the system of partial differential equations in terms of the electromagnetic potentials Ψ_e and Ψ_m. These potentials are derived with respect to the Killing vector X. In the same way we may use the electromagnetic potentials with respect to the Killing vector Y denoted respectively as Ψ_e^Y and Ψ_m^Y. Moreover, we may also use a mixed pair of potentials, say Ψ_e and Ψ_e^Y. In general, the transition between different pairs of potentials may be performed by the following formulas

$$\partial_\xi \Psi_m^Y = -\xi \, e^{-2\varphi-2p} \, \partial_z \Psi_e,$$

$$\partial_z \Psi_m^Y = -\xi \, e^{-2\varphi-2p} \, \partial_\xi \Psi_e,$$

$$\partial_z \Psi_e^Y = -\xi \, e^{2\varphi-2p} \, \partial_\xi \Psi_m,$$

$$\partial_\xi \Psi_e^Y = -\xi \, e^{2\varphi-2p} \, \partial_z \Psi_m.$$

For the reader's convenience we have presented in the Appendix the system (8) - (14) written in terms of the mixed pair $\Psi_e = \omega$ and $\Psi_e^Y = \chi$.

Let us go back again to the system of nonlinear partial differential equations (8) - (14). We have a complicated system of coupled nonlinear partial differential equations and it seems that finding all its solutions is a hopeless task. Nevertheless, as we will see a large enough class of solutions can be found.

B. Solution generating method

Here we consider the case in which one of the electromagnetic potentials vanishes. For definiteness we take $\Psi_e = 0$. Let us introduce the new potentials $u = p - \varphi$, $\phi = p + \varphi$, $\Psi_m^S = \sqrt{2}\Psi_m$ and $h^S = 2h$. Then the system (8) - (14) may be rewritten in terms of the new variables as follows

$$\partial_\xi^2 \phi + \frac{1}{\xi}\partial_\xi \phi - \partial_z^2 \phi = 0,$$

$$\partial_\xi^2 u + \frac{1}{\xi}\partial_\xi u - \partial_z^2 u = e^{-2u}\left((\partial_z \Psi_m^S)^2 - (\partial_\xi \Psi_m^S)^2\right),$$

$$\partial_\xi \left(\xi e^{-2u}\partial_\xi \Psi_m^S\right) - \partial_z \left(\xi e^{-2u}\partial_z \Psi_m^S\right) = 0, \qquad (15)$$

$$\frac{1}{\xi}\partial_\xi h^S = (\partial_\xi \phi)^2 + (\partial_z \phi)^2 + (\partial_\xi p^S)^2 + (\partial_z p^S)^2 +$$
$$e^{-2u}\left((\partial_\xi \Psi_m^S)^2 + (\partial_z \Psi_m^S)^2\right),$$

$$\frac{1}{\xi}\partial_z h^S = 2\partial_\xi\phi\partial_z\phi + 2\partial_\xi u\partial_z u + 2e^{-2u}\partial_\xi\Psi_m^S\partial_z\Psi_m^S.$$

It is not difficult to recognize that the system (15) coincides with the corresponding one for the Einstein-Maxwell gravity with a minimally coupled scalar field ϕ for the metric

$$ds^2 = e^{2h^S-2u}(dz^2 - d\xi^2) + e^{2u}dx^2 + e^{-2u}\xi^2 dy^2. \quad (16)$$

Here Ψ_m^S is the corresponding non-vanishing electromagnetic potential in the case of a minimally coupled scalar field.

In this way we have proven the following proposition

Proposition B.1 *Let u, ϕ, h^S and Ψ_m^S be a solution to the Einstein-Maxwell equations with a minimally coupled scalar field for the metric (16). Then $p = \frac{1}{2}(u + \phi)$, $\varphi = \frac{1}{2}(\phi - u)$, $h = \frac{1}{2}h^S$ and $\Psi_m = \frac{1}{\sqrt{2}}\Psi_m^S$ form a solution to the equations of EMD gravity for the metric (7).*

This result allows us to generate exact solutions to the EMD gravity equations for the metric (7) directly from known solutions of the Einstein - Maxwell equations with a minimally coupled scalar field.

Before going further we shall make some comments. In the case when the space-time admits two space-like Killing symmetries and one of the electromagnetic potentials is taken to vanish, it may be shown that the equations governing the transverse part of the gravitational field, dilaton and non-vanishing electromagnetic potential are written in the compact matrix form

$$\partial_\xi\left(\xi S^{-1}\partial_\xi S\right) - \partial_z\left(\xi S^{-1}\partial_z S\right) = 0. \quad (17)$$

The equation (17) is just the chiral equation. There are powerful solitonic techniques for solving this equation (see [23]). Although, the study of cosmological soliton solutions in EMD gravity seems to be very interesting, we will not consider them in the preset work.

IV. EXAMPLES OF EXACT SOLUTIONS

A. Homogeneous solutions

Homogeneous solutions are obtained when we have dependence only on the "time - coordinate" ξ. In this case the differential constraint reduces to $\partial_\xi\Psi_e\partial_\xi\Psi_m = 0$. Thus we may choose one of the potentials to vanish. Therefore we may apply the proposition from the previous section to generate homogeneous solutions starting from the corresponding homogeneous solutions for the case of a minimally coupled scalar field. In our notations the homogeneous solutions for a minimally coupled scalar field are [13]

$$\phi = \phi_0 + \beta_0\ln(\xi), \quad (18)$$

$$u = \ln\left(e^{\frac{\phi_1}{2}}\xi^{1+\frac{\alpha_0}{2}} + e^{\frac{-\phi_1}{2}}\xi^{1-\frac{\alpha_0}{2}}\right), \quad (19)$$

$$\Psi_e^{SY} = \frac{1}{2}\tanh(\frac{\phi_1}{2} + \frac{\alpha_0}{2}\ln(\xi)), \quad (20)$$

where ϕ_0, ϕ_1, α_0 and β_0 are arbitrary constants.

The above seed solution (18) - (20) gives the following homogeneous solution to the EMD equations. The potential Ψ_e^Y is

$$\Psi_e^Y = \frac{1}{2\sqrt{2}}\tanh(\frac{\phi_1}{2} + \frac{\alpha_0}{2}\ln(\xi)).$$

The metric has the form

$$ds^2 = A^2(\xi)(dz^2 - d\xi^2) + B^2(\xi)dx^2 + C^2(\xi)dy^2$$

where

$$A^2(\xi) = e^{\gamma_0-\phi_0}\xi^{\frac{\alpha_0^2}{4}+\beta_0^2-\beta_0}\left(e^{\frac{\phi_1}{2}}\xi^{\frac{\alpha_0}{2}} + e^{\frac{-\phi_1}{2}}\xi^{-\frac{\alpha_0}{2}}\right),$$

$$B^2(\xi) = e^{\phi_0}\xi^{1+\beta_0}\left(e^{\frac{\phi_1}{2}}\xi^{\frac{\alpha_0}{2}} + e^{\frac{-\phi_1}{2}}\xi^{-\frac{\alpha_0}{2}}\right),$$

$$C^2(\xi) = e^{-\phi_0}\xi^{1-\beta_0}\left(e^{\frac{\phi_1}{2}}\xi^{\frac{\alpha_0}{2}} + e^{\frac{-\phi_1}{2}}\xi^{-\frac{\alpha_0}{2}}\right)^{-1}.$$

Here γ_0 is a constant.

The asymptotic behaviour of the expansion factors and the dilaton field in the limit $\xi \to 0$ is as follows

$$A^2(\xi) \sim \xi^{\frac{\alpha_0^2}{4}+\beta_0^2-\beta_0-\frac{|\alpha_0|}{2}},$$

$$B^2(\xi) \sim \xi^{1+\beta_0-\frac{|\alpha_0|}{2}},$$

$$C^2(\xi) \sim \xi^{1-\beta_0+\frac{|\alpha_0|}{2}},$$

$$\varphi \sim -\frac{1}{2}(1 - \beta_0 - \frac{1}{2}\mid\alpha_0\mid)\ln(\xi).$$

The asymptotic form of the expansion factors and the dilaton field in the limit $\xi \gg 1$ is

$$A^2(\xi) \sim \xi^{\frac{\alpha_0^2}{4}+\beta_0^2-\beta_0+\frac{|\alpha_0|}{2}},$$

$$B^2(\xi) \sim \xi^{1+\beta_0+\frac{|\alpha_0|}{2}},$$

$$C^2(\xi) \sim \xi^{1-\beta_0-\frac{|\alpha_0|}{2}},$$

$$\varphi \sim -\frac{1}{2}(1 - \beta_0 + \frac{1}{2} \mid \alpha_0 \mid) \ln(\xi).$$

Introducing the scynhroneous time $d\tau = A(\xi)d\xi$, the line element in the limits $\xi \to 0$ and $\xi \gg 1$ can be written in the Kasner form

$$ds^2 \sim -d\tau^2 + \tau^{2p_1}dx^2 + \tau^{2p_2}dy^2 + \tau^{2p_3}dz^2$$

where the Kasner exponents are defined by

$$p_1 = \frac{1 + \beta_0 \mp \frac{1}{2} \mid \alpha_0 \mid}{\frac{1}{4}(1\mp \mid \alpha_0 \mid)^2 + (\frac{1}{2} - \beta_0)^2 + \frac{3}{2}} \,,$$

$$p_2 = \frac{1 - \beta_0 \pm \frac{1}{2} \mid \alpha_0 \mid}{\frac{1}{4}(1\mp \mid \alpha_0 \mid)^2 + (\frac{1}{2} - \beta_0)^2 + \frac{3}{2}} \,,$$

$$p_3 = \frac{\frac{1}{4}(1\mp \mid \alpha_0 \mid)^2 + (\frac{1}{2} - \beta_0)^2 - \frac{1}{2}}{\frac{1}{4}(1\mp \mid \alpha_0 \mid)^2 + (\frac{1}{2} - \beta_0)^2 + \frac{3}{2}} \,,$$

as the sing "$-$" refers to the limit $\xi \to 0$ while the sign "$+$" is for the limit $\xi \gg 1$.

The dilaton filed is given by

$$\varphi \sim \sigma \ln(\tau)$$

where

$$\sigma = -\frac{1 - \beta_0 \mp \frac{1}{2}\alpha_0}{\frac{1}{4}(1\mp \mid \alpha_0 \mid)^2 + (\frac{1}{2} - \beta_0)^2 + \frac{3}{2}} \,.$$

The parameters p_1, p_2 and p_3 satisfy the Belinskii - Khalatnikov relations

$$p_1 + p_2 + p_3 = 1,$$
$$p_1^2 + p_2^2 + p_3^2 = 1 - 2\sigma^2 \,.$$

B. Inhomogeneous solutions

Here the proposition (B.1) will be applied again, this time to generate inhomogeneous exact solutions to the EMD equations. As a seed family of solutions we take the Charach family of solutions describing inhomogeneous cosmologies with minimally coupled scalar and electromagnetic fields and with $S^1 \times S^1 \times S^1$ topology of the spatial sections. The Charach family of inhomogeneous solutions and their asymptotics in the two limiting cases are presented in the Appendix.

According to the proposition (B.1) the inhomogeneous EMD metric is given by

$$ds^2 = e^{2(h-p)}(dz^2 - dt^2) + e^{2p}dx^2 + \xi^2 e^{-2p}dy^2$$

where

$$2p = \ln(2\xi \cosh(\frac{1}{2}\tilde{\phi})) + \phi,$$

$$h = \frac{1}{2}\ln(\xi) + \ln(2\cosh(\frac{1}{2}\tilde{\phi})) +$$

$$\frac{1}{4}F(\tilde{\phi}_0, \alpha_0, A_n, B_n; \xi, z) + F(\phi_0, \beta_0, C_n, D_n; \xi, z).$$

The dilaton field is

$$\varphi = \frac{1}{2}\phi - \frac{1}{2}\ln\left(2\xi \cosh(\frac{1}{2}\tilde{\phi})\right).$$

The inhomogeneous cosmological solutions introduce characteristic length scales. In fact each normal mode has its own characteristic scale. The horizon distance in the "z" direction is given by $ds^2 \mid_{x,y} = 0$ and hence

$$\delta z = \int_0^\xi d\xi = \xi.$$

In this way $n\xi$ can be viewed as the ratio of the horizon distance in the "z" direction to the coordinate wavelength λ_n i.e. $n\xi = \frac{\delta z}{\lambda_n}$.

There are two limiting cases to be considered. The first case is when the wavelength is much larger than the horizon scale ($n\xi \ll 1$). The second case is when the wavelength is much less than the horizon scale ($n\xi \gg 1$).

In the first case ($n\xi \ll 1$) the asymptotic form of the EMD metric is

$$ds^2 = A^2(\xi, z)(dz^2 - d\xi^2) + B^2(\xi, z)dx^2 + C^2(\xi, z)dy^2$$

where

$$A^2(\xi, z) =$$
$$e^{\gamma(z)}\xi^{\frac{1}{4}\alpha^2(z)+\beta^2(z)+\beta(z)}\left(e^{\frac{1}{2}\tilde{\phi}_*(z)}\xi^{\frac{1}{2}\alpha(z)} + e^{-\frac{1}{2}\tilde{\phi}_*(z)}\xi^{-\frac{1}{2}\alpha(z)}\right),$$

$$B^2(\xi, z) =$$
$$e^{\tilde{\phi}_*(z)} \xi^{1+\beta(z)} \left(e^{\frac{1}{2}\tilde{\phi}_*(z)} \xi^{\frac{1}{2}\alpha(z)} + e^{-\frac{1}{2}\tilde{\phi}_*(z)} \xi^{-\frac{1}{2}\alpha(z)} \right),$$

$$C^2(\xi, z) =$$
$$e^{-\tilde{\phi}_*(z)} \xi^{1-\beta(z)} \left(e^{\frac{1}{2}\tilde{\phi}_*(z)} \xi^{\frac{1}{2}\alpha(z)} + e^{-\frac{1}{2}\tilde{\phi}_*(z)} \xi^{-\frac{1}{2}\alpha(z)} \right)^{-1}.$$

In the limit $n\xi \ll 1$ the asymptotic form of the dilaton is

$$\varphi \sim \frac{1}{2} \left(\phi_*(z) + sign(\alpha(z)) \tilde{\phi}_*(z) \right) -$$
$$\frac{1}{2} \left(1 - \beta(z) - \frac{1}{2} \mid \alpha(z) \mid \right) \ln(\xi).$$

In order to discuss the cosmological solutions near the singularity we have to consider homogeneous limit for ξ approaching zero. Introducing the proper time, in the homogeneous limit, by

$$\tau = \int A(\xi) d\xi,$$

we find that the metric has the Kasner form

$$g_{\mu\nu} \sim (-1, \tau^{2p_1}, \tau^{2p_2}, \tau^{2p_3}).$$

Here the Kasner indexes are spatially varying

$$p_1(z) = \frac{1 + \beta(z) \mp \frac{1}{2} \mid \alpha(z) \mid}{\frac{1}{4}(1 \mp \mid \alpha(z) \mid)^2 + (\frac{1}{2} - \beta(z))^2 + \frac{3}{2}},$$

$$p_2(z) = \frac{1 - \beta(z) \pm \frac{1}{2} \mid \alpha(z) \mid}{\frac{1}{4}(1 \mp \mid \alpha(z) \mid)^2 + (\frac{1}{2} - \beta(z))^2 + \frac{3}{2}},$$

$$p_3(z) = \frac{\frac{1}{4}(1 \mp \mid \alpha(z) \mid)^2 + (\frac{1}{2} - \beta(z))^2 - \frac{1}{2}}{\frac{1}{4}(1 \mp \mid \alpha(z) \mid)^2 + (\frac{1}{2} - \beta(z))^2 + \frac{3}{2}}$$

and satisfy the Belinski- Khalatnikov relations

$$p_1(z) + p_2(z) + p_3(z) = 1,$$

$$p_1^2(z) + p_2^2(z) + p_3^2(z) = 1 - 2\sigma^2(z),$$

where

$$\sigma = -\frac{1 - \beta(z) \mp \frac{1}{2}\alpha(z)}{\frac{1}{4}(1 \mp \mid \alpha(z) \mid)^2 + (\frac{1}{2} - \beta(z))^2 + \frac{3}{2}}.$$

In the second case when $n\xi \gg 1$ the asymptotic form of the EMD metric is as follows.

The first sub-case is for $\alpha_0 = 0$. Then the EMD metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where

$$\eta = diag \left(-2\xi^{\beta_0^2 - \beta_0} e^{\frac{1}{2}K\xi}, 2\xi^{1+\beta_0}, \right.$$
$$\left. \frac{1}{2}\xi^{1-\beta_0}, 2\xi^{\beta_0^2 - \beta_0} e^{\frac{1}{2}K\xi} \right)$$

and

$$h = diag \left(0, \frac{2}{\sqrt{\xi}} \xi^{1+\beta_0} H(\xi, z), -\frac{2}{\sqrt{\xi}} \xi^{1-\beta_0} H(\xi, z), 0 \right).$$

When $\alpha_0 \neq 0$ we have

$$\eta = diag \left(-\xi^{\frac{1}{4}(1+\alpha_0)^2 + (\beta_0 - \frac{1}{2})^2 - \frac{1}{2}} e^{\frac{1}{2}K\xi}, \xi^{1 + \frac{1}{2}|\alpha_0| + \beta_0}, \right.$$
$$\left. \xi^{1 - \frac{1}{2}|\alpha_0| - \beta_0}, \xi^{\frac{1}{4}(1+\alpha_0)^2 + (\beta_0 - \frac{1}{2})^2 - \frac{1}{2}} e^{\frac{1}{2}K\xi} \right),$$

$$h = diag \left(0, \frac{\xi^{1 + \frac{1}{2}|\alpha_0| + \beta_0}}{\sqrt{\xi}} (H(\xi, z) + \frac{1}{2} sign(\alpha_0)\tilde{H}(\xi, z)), \right.$$
$$\left. -\frac{\xi^{1 - \frac{1}{2}|\alpha_0| - \beta_0}}{\sqrt{\xi}} (H(\xi, z) + \frac{1}{2} sign(\alpha_0)\tilde{H}(\xi, z)), 0 \right).$$

The asymptotic form of the dilaton in the limit $n\xi \gg 1$, for both $\alpha_0 = 0$ and $\alpha_0 \neq 0$, is given by

$$\varphi \sim -\frac{1}{2} \left(1 - \beta_0 + \frac{1}{2} \mid \alpha_0 \mid \right) \ln(\xi) +$$
$$\frac{1}{\sqrt{\xi}} \left(H(\xi, z) + \frac{1}{2} sign(\alpha_0)\tilde{H}(\xi, z) \right).$$

The explicit form of the solutions and their asymptotics allow us to make some conclusions. The obtained solutions fall in the category of asymptotic velocity-term dominated space-times. When the singularity is approached the spatial derivatives become negligible with comparison to the time derivatives. Sufficiently close to the singularity the evolution at different spatial points is decoupled and the metric is locally Kasner with spatially dependent Kasner indices satisfying the Belinskii-Khalatnikov relations.

The late evolution of the exact cosmological solutions is described by a homogeneous, anisotropic universe with gravitational, scalar (dilatonic) and electromagnetic waves. The non-minimal dilaton-electromagnetic exponential coupling influences mainly the homogeneous background universe rather the scalar and electromagnetic waves on that background. The former may be considered as minimally coupled scalar and electromagnetic waves up to higher orders of $\frac{1}{\xi}$.

V. CONCLUSION

In this paper we have shown that it is possible to find exact inhomogeneous and anisotropic cosmological solutions of low energy string theory containing non-minimally interacting dilaton and Maxwell fields - Einstein-Maxwell -dilaton gravity. First we have considered space-times admitting one hyper-surface orthogonal Killing vector. It has been shown that in the case of one vanishing electromagnetic potential the dimensionally reduced equations possess a group of global symmetry $GL(2, R)$. We have described an algorithm for generating exact cosmological EMD solutions starting from vacuum and dilaton-vacuum backgrounds by employing the nonlinear action of the global symmetry group. This algorithm is especially useful for generating exact cosmological EMD solutions with only one Killing vector, starting with G_1-dilaton-vacuum background. Another method which permits to construct exact EMD solutions starting from solutions of the pure Einstein equations has been briefly discussed, too.

In the case when the space-time admits two commuting, hyper-surface orthogonal Killing vectors we have given a method which allows exact inhomogeneous cosmological solutions to the EMD equations to be generated from the corresponding solutions of the Einstein-Maxwell equations with a minimally coupled scalar field. Using this method, as a particular case, we have obtained exact cosmological homogeneous and inhomogeneous solutions to the EMD equations. The initial evolution described by these solutions is of a spatially varying Kasner form. The intermediate stage of evolution occurs when the characteristic scales of the inhomogeneities approach the scale of the particle horizon. This stage is characterized by strongly interacting non-linear gravitational, dilatonic and electromagnetic waves. The late evolution of the cosmological solutions is described by a background homogeneous and anisotropic universe filled with weakly interacting gravitational, dilatonic and electromagnetic waves.

The cosmological solutions found in the present work fall in the category of AVTD space-times. Near the singularity the dynamics at different spatial points decouples and the metric has a spatially varying Kasner form. Therefore, the non-minimal coupling of the dilaton to the Maxwell field does not change the nature of the singularity. The solutions are sufficiently generic and therefore we should expect that the T^3 space-times in EMD gravity have AVTD singularities in general.

Finally we believe that the present paper will be a sufficiently good background for studying similar problems in the more-general case of Enstein-Maxwell-dilaton-axion gravity.

APPENDIX A: REDUCED EMD SYSTEM IN TERMS OF THE POTENTIALS $\Psi_E = \omega$ AND $\Psi_E^Y = \chi$

$$\partial_\xi^2 p + \frac{1}{\xi}\partial_\xi p - \partial_z^2 p = e^{-2\varphi-2p}\left((\partial_z\omega)^2 - (\partial_\xi\omega)^2\right)$$
$$+ \frac{1}{\xi^2}e^{-2\varphi+2p}\left((\partial_\xi\chi)^2 - (\partial_z\chi)^2\right)$$

$$\partial_\xi^2\varphi + \frac{1}{\xi}\partial_\xi\varphi - \partial_z^2\varphi = e^{-2\varphi-2p}\left((\partial_z\omega)^2 - (\partial_\xi\omega)^2\right)$$
$$- \frac{1}{\xi^2}e^{-2\varphi+2p}\left((\partial_\xi\chi)^2 - (\partial_z\chi)^2\right)$$

$$\partial_\xi\omega\,\partial_\xi\chi = \partial_z\omega\,\partial_z\chi$$

$$\partial_\xi^2\omega + \frac{1}{\xi}\partial_\xi\omega - \partial_z^2\omega =$$
$$2(\partial_\xi\varphi + \partial_\xi p)\partial_\xi\omega - 2(\partial_z\varphi + \partial_z p)\partial_z\omega$$

$$\partial_\xi^2\chi - \frac{1}{\xi}\partial_\xi\chi - \partial_z^2\chi =$$
$$2(\partial_\xi\varphi - \partial_\xi p)\partial_\xi\chi - 2(\partial_z\varphi - \partial_z p)\partial_z\chi$$

$$\frac{1}{\xi}\partial_\xi h = (\partial_\xi\varphi)^2 + (\partial_z\varphi)^2 + (\partial_\xi p)^2 + (\partial_z p)^2 +$$
$$e^{-2\varphi-2p}\left((\partial_\xi\omega)^2 + (\partial_\xi\omega)^2\right) +$$
$$\frac{1}{\xi^2}e^{-2\varphi+2p}\left((\partial_\xi\chi)^2 + (\partial_\xi\chi)^2\right)$$

$$\frac{1}{\xi}\partial_z h = 2\partial_\xi\varphi\partial_z\varphi + 2\partial_\xi p\partial_z p +$$
$$2e^{-2\varphi-2p}\partial_\xi\omega\partial_z\omega + 2e^{-2\varphi+2p}\partial_\xi\chi\partial_z\chi$$

APPENDIX B: CHARACH FAMILY OF EXACT SOLUTIONS

For the reader's convenience we present here the Charach family of solutions written in our notations.

The minimally coupled scalar filed is given by

$$\phi = \phi_0 + \beta_0 \ln(\xi) +$$
$$\sum_{n=1}^{\infty} (C_n J_0(n\xi) + D_n N_0(n\xi)) \cos(n(z - z_n)).$$

Here ϕ_0, β_0, C_n, D_n are arbitrary constants and $J_0(.)$ and $N_0(.)$ are respectively the Bessel and Neumann functions.

The metric function u and the non-vanishing electromagnetic potential Ψ_e^{SY} are correspondingly

$$u = \ln\left(2\xi \cosh(\frac{1}{2}\tilde{\phi})\right),$$

$$\Psi_e^{SY} = const + \frac{1}{2}\tanh(\frac{1}{2}\tilde{\phi})$$

where $\tilde{\phi}$ is an auxiliary scalar field which is of the form

$$\tilde{\phi} = \tilde{\phi}_0 + \alpha_0 \ln(\xi) +$$
$$\sum_{n=1}^{\infty} (A_n J_0(n\xi) + B_n N_0(n\xi)) \cos(n(z - z_n)).$$

The longitudinal part of the gravitational field is

$$h^S = \ln(\xi) + 2\ln(2\cosh(\frac{1}{2}\tilde{\phi})) +$$
$$\frac{1}{2}F(\tilde{\phi}_0, \alpha_0, A_n, B_n; \xi, z) + 2F(\phi_0, \beta_0, C_n, D_n; \xi, z)$$

where F is a solution to the system

$$\frac{1}{\xi}\partial_\xi F = \frac{1}{2}((\partial_\xi \phi)^2 + (\partial_z \phi)^2),$$
$$\frac{1}{\xi}\partial_z F = \partial_\xi \phi \partial_z \phi.$$

For more details see [13].

In the limit $\xi \to 0$ the minimally coupled scalar field and the auxiliary field behave as

$$\phi \sim \phi_*(z) + \beta(z)\ln(\xi),$$
$$\tilde{\phi} \sim \tilde{\phi}_*(z) + \alpha(z)\ln(\xi)$$

where

$$\alpha(z) = \alpha_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} B_n \cos(n(z - z_n)),$$

$$\beta(z) = \beta_0 + \frac{2}{\pi}\sum_{n=1}^{\infty} D_n \cos(n(z - z_n))$$

and

$$\phi_*(z) = \phi_0 +$$
$$\sum_{n=1}^{\infty}\left(C_n + \frac{2}{\pi}D_n(\gamma + \ln(\frac{n}{2}))\right)\cos(n(z - z_n)),$$

$$\tilde{\phi}_*(z) = \tilde{\phi}_0 +$$
$$\sum_{n=1}^{\infty}\left(A_n + \frac{2}{\pi}B_n(\gamma + \ln(\frac{n}{2}))\right)\cos(n(z - z_n)).$$

Here γ is the Euler constant.

The asymptotic form of the Charach metric can be written as

$$ds^2 = A_s^2(\xi, z)(dz^2 - d\xi^2) + B_S^2(\xi, z)dx^2 + C_S^2(\xi, z)dy^2$$

where

$$A_S(\xi, z) =$$
$$e^{\gamma(z)}\xi^{\frac{1}{4}\alpha^2(z)+\beta^2(z)}\left(e^{\frac{1}{2}\tilde{\phi}_*(z)}\xi^{\frac{1}{2}\alpha(z)} + e^{-\frac{1}{2}\tilde{\phi}_*(z)}\xi^{-\frac{1}{2}\alpha(z)}\right),$$

$$B_S(\xi, z) = \xi\left(e^{\frac{1}{2}\tilde{\phi}_*(z)}\xi^{\frac{1}{2}\alpha(z)} + e^{-\frac{1}{2}\tilde{\phi}_*(z)}\xi^{-\frac{1}{2}\alpha(z)}\right),$$

$$C_S(\xi, z) = \left(e^{\frac{1}{2}\tilde{\phi}_*(z)}\xi^{\frac{1}{2}\alpha(z)} + e^{-\frac{1}{2}\tilde{\phi}_*(z)}\xi^{-\frac{1}{2}\alpha(z)}\right)^{-1}.$$

The asymptotic behaviour of the scalar fields in the Charach solution in the high frequency regime ($n\xi \gg 1$) is

$$\phi \sim \phi_0 + \alpha_0 \ln\xi + \xi^{-\frac{1}{2}}H(\xi, z),$$
$$\tilde{\phi} \sim \tilde{\phi}_0 + \beta_0 \ln\xi + \xi^{-\frac{1}{2}}\tilde{H}(\xi, z)$$

where

$$\tilde{H}(\xi, z) = Re\sum_{n=-\infty,\neq 0}^{\infty}(A_{|n|} - iB_{|n|})\frac{e^{-i(z_n - \frac{\pi}{4})}}{\sqrt{2\pi|n|}}e^{i(|n|\xi + nz)},$$

$$H(\xi, z) = Re\sum_{n=-\infty,\neq 0}^{\infty}(C_{|n|} - iD_{|n|})\frac{e^{-i(z_n - \frac{\pi}{4})}}{\sqrt{2\pi|n|}}e^{i(|n|\xi + nz)}.$$

The functions $H(\xi, z)$, $\tilde{H}(\xi, z)$ satisfy the D'Alembert equations

$$\partial_\xi^2 H(\xi, z) - \partial_z^2 H(\xi, z) = 0,$$
$$\partial_\xi^2 \tilde{H}(\xi, z) - \partial_z^2 \tilde{H}(\xi, z) = 0.$$

When $\alpha_0 = 0$ the asymptotic form of the metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where

$$\eta = diag\left(-4\xi^{2\beta_0^2}e^{K\xi}, 4\xi^2, \frac{1}{4}, 4\xi^{2\beta_0^2}e^{K\xi}\right)$$

and

$$h = diag\left(0, 4\xi^2 H(\xi,z)/4\xi, -\frac{1}{4}H^2(\xi,z)/4\xi\right).$$

The constant K is given by

$$K = \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2 + 4(C_n^2 + D_n^2)\right).$$

In the case when $\alpha_0 \neq 0$ the asymptotic form of the metric is

$$\eta = diag\left(-\xi^{2\beta_0^2+\frac{1}{2}|\alpha_0|+\frac{1}{2}|\alpha_0|^2}e^{K\xi}, \xi^{|\alpha_0|+2},\right.$$
$$\left.\xi^{-|\alpha_0|}, \xi^{2\beta_0^2+\frac{1}{2}|\alpha_0|+\frac{1}{2}|\alpha_0|^2}e^{K\xi}\right)$$

and

$$h = diag\left(0, \pm\xi^{|\alpha_0|+2}H(\xi,z)/\sqrt{\xi},\right.$$
$$\left.\mp\xi^{-|\alpha_0|}H(\xi,z)/\sqrt{\xi}, 0\right).$$

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